Characterization of rearrangement invariant spaces with fixed points for the Hardy–Littlewood maximal operator

Joaquim Martín[§] and Javier Soria[¶]

Abstract. We characterize the rearrangement invariant spaces for which there exists a non-constant fixed point, for the Hardy–Littlewood maximal operator (the case for the spaces $L^p(\mathbb{R}^n)$ was first considered in [7]). The main result that we prove is that the space $L^{\frac{n}{n-2},\infty}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ is minimal among those having this property.

1 Introduction

The centered Hardy–Littlewood maximal operator $\mathcal{M}$ is defined on the Lebesgue space $L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$\mathcal{M}f(x)=\sup_{r>0}\frac{1}{|B_r|}\int_{B_r}|f(x-y)|\,dy,$$

where $|B_r|$ denotes the measure of the Euclidean ball B_r centered at the origin of $\mathbb{R}^n$.

In this paper we study the existence of non–constant fixed points of the maximal operator $\mathcal{M}$ (i.e., $\mathcal{M}f=f$) in the framework of the rearrangement invariant (r.i.) functions spaces (see Section 2 below). We will use some of the estimates proved in [7], where the case $L^p(\mathbb{R}^n)$ was studied, and show that they can be sharpened to obtain all the rearrangement invariant norms with this property (in particular we extend Korry's result to the end point case $p=n/(n-2)$, where the weak–type spaces have to considered.) The main argument behind this problem is the existence of a minimal space $L^{\frac{n}{n-2},\infty}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ contained in all the r.i. spaces with the fixed point property.

2 Background on Rearrangement Invariant Spaces

Since we work in the context of rearrangement invariant spaces it will be convenient to start by reviewing some basic definitions about these spaces.

A rearrangement invariant space $X=X(\mathbb{R}^n)$ (r.i. space) is a Banach function space on $\mathbb{R}^n$ endowed with a norm $\|\cdot\|_{X(\mathbb{R}^n)}$ such that

$$\|f\|_{X(\mathbb{R}^n)}=\|g\|_{X(\mathbb{R}^n)}$$

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whenever $f^* = g^*$. Here f^* stands for the non-increasing rearrangement of f, i.e., the non-increasing, right-continuous function on $[0, \infty)$ equimeasurable with f.

An r.i. space $X(\mathbb{R}^n)$ has a representation as a function space on $\bar{X}(0, \infty)$ such that

$$\|f\|_{X(\mathbb{R}^n)} = \|f^*\|_{\bar{X}(0,\infty)} \,.$$

Any r.i. space is characterized by its **fundamental function**

$$\phi_X(s) = \|\chi_E\|_{X(\mathbb{R}^n)}$$

(here E is any subset of $\mathbb{R}^n$ with $|E| = s$) and the **fundamental indices**

$$\overline{\beta}_X = \inf_{s>1} \frac{\log M_X(s)}{\log s} \quad \text{and} \quad \underline{\beta}_X = \sup_{s<1} \frac{\log M_X(s)}{\log s},$$

where

$$M_X(s) = \sup_{t>0} \frac{\phi_X(ts)}{\phi_X(t)}, \ \ s > 0.$$

It is well known that

$$0 \leq \underline{\beta}_X \leq \overline{\beta}_X \leq 1.$$

(We refer the reader to [2] for further information about r.i. spaces.)

3 Main result

Before formulating our main result, it will be convenient to start with the following remarks (see [7]):

Remark 3.1 By the Lebesgue's differentiation theorem one easily obtains that

$$|f(x)| \leq \mathcal{M}f(x) \text{ a.e. } x \in \mathbb{R}^n,$$

thus f is a fixed point of $\mathcal{M}$, if and only if f is positive and

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy \leq f(x) \text{ a.e. } x \in \mathbb{R}^n,$$

or equivalently f is a positive **super–harmonic** function (i.e. $\triangle f \leq 0$, where $\triangle$ is the Laplacian operator).

Remark 3.2 If f is a non-constant fixed point of $\mathcal{M}$, and $\varphi \geq 0$ belongs to the Schawrtz class $\mathcal{S}(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, then the function $f_t(x) = (f * \varphi_t)(x)$, with $\varphi_t(x) = t^{-n}\varphi(x/t)$ is also a non-constant fixed point of $\mathcal{M}$ which belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$ (notice that using the Lebesgue differentiation theorem, there exists some $t > 0$ such that f_t is non-constant, since f is non-constant). In particular if $X(\mathbb{R}^n)$ is an r.i. space and $f \in X(\mathbb{R}^n)$ is a non-constant fixed point of $\mathcal{M}$, since $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ we get that $f_t \in X(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ is a non-constant fixed point of $\mathcal{M}$.

Remark 3.3 Using the theory of weighted inequalities for $\mathcal{M}$ (see [6]), if $\mathcal{M}f = f$, in particular $f \in A_1$ (the Muckenhoupt weight class), and hence $f(x)\,dx$ defines a doubling measure. Hence, $f \notin L^1(\mathbb{R}^n)$. Also, using the previous remark we see that if $f \in L^p(\mathbb{R}^n)$ is a fixed point, then $f \in L^q(\mathbb{R}^n)$, for all $p \le q \le \infty$.

Definition 3.4 *Given an r.i. space $X(\mathbb{R}^n)$, we define*

$$D_{I_2}(X(\mathbb{R}^n)) = \left\{ f \in L^0(\mathbb{R}^n) : \|I_2 f\|_{X(\mathbb{R}^n)} < \infty \right\},$$

where I_2 is the Riesz potential,

$$(I_2 f)(x) = \int_{\mathbb{R}^n} |x-y|^{2-n} f(y)dy.$$

It is not hard to see that the space $D_{I_2}(X(\mathbb{R}^n))$ is either trivial or is the largest r.i. space which is mapped by I_2 into $X(\mathbb{R}^n)$, and is also related with the theory of the optimal Sobolev embeddings (see [4] and the references quoted therein).

Theorem 3.5 *Let $X(\mathbb{R}^n)$ be an r.i. space. The following statements are equivalent:*

1. *There is a non-constant fixed point $f \in X(\mathbb{R}^n)$ of $\mathcal{M}$*
2. *$n \ge 3$ and $|x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n)$.*
3. *$n \ge 3$ and $\chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t) \in \bar{X}(0,\infty)$.*
4. *$n \ge 3$ and $(L^{\frac{n}{n-2},\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \subset X(\mathbb{R}^n)$.*
5. *$n \ge 3$ and $D_{I_2}(X(\mathbb{R}^n)) \neq \{0\}$.*

Proof. $(1 \to 2)$ Since if $n = 1$ or $n = 2$, the only positive super–harmonic functions are the constant functions (see [8, Remark 1, p. 210]), necessarily $n \ge 3$. Moreover, it is proved in [7] that, if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a non-constant fixed point of $\mathcal{M}$, then

$$f(x) \ge c\,|x|^{2-n} \chi_{\{x:|x|>1\}}(x).$$

Since $f \in X(\mathbb{R}^n)$, then $|x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n)$.

$(2 \to 3)$ Since if $|x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n)$, then

$$F(x) = \chi_{\{x:|x|\le 1\}}(x) + |x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n).$$

An easy computation shows that

$$F^*(t) \simeq \chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t).$$

$(3 \to 4)$ Since $f \in (L^{\frac{n}{n-2},\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ if and only if

$$\sup_{t>0} f^*(t)W(t) < \infty,$$

where $W(t) = \max(1, t^{1-2/n})$, we have that

$$f^*(t) \leq \|f\|_{L^{\frac{n}{n-2},\infty}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)} W^{-1}(t)$$

and since $W^{-1}(t) = \chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)} \in \bar{X}(0,\infty)$ we have that

$$\|f\|_{X(\mathbb{R}^n)} = \|f^*\|_{\bar{X}(0,\infty)} \leq c\,\|f\|_{L^{\frac{n}{n-2},\infty}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)}$$

with $c = \|W^{-1}\|_{\bar{X}(0,\infty)}$.

$(4 \to 5)$ Since (see [9] and [1])

$$(I_2 f)^*(t) \leq c_1 \left(t^{2/n-1} \int_0^t f^*(s)ds + \int_t^\infty f^*(s)s^{2/n-1}ds \right) \leq c_2 \left(I_2 f^0\right)^*(t)$$

where $f^0(x) = f^*(c_n\,|x|^n)$, c_n = measure of the unit ball in $\mathbb{R}^n$. (Observe that $(f^0)^* = f^*$). Rewriting the middle term in the above inequalities, using Fubini's theorem, we get

$$(I_2 f)^*(t) \leq d_1 \left(\frac{n}{n-2} \int_t^\infty f^{**}(s)s^{2/n-1}ds \right) \leq d_2 \left(I_2 f^0\right)^*(t),$$

where $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds$. Thus, $f \in D_{I_2}(X(\mathbb{R}^n))$ if and only if

$$\left\| \int_t^\infty f^{**}(s)s^{2/n-1}ds \right\|_{\bar{X}(0,\infty)} < \infty. \tag{1}$$

Since

$$F(t) = \int_t^\infty \chi_{[0,1]}^{**}(s)s^{2/n-1}ds = c(\chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t))$$

is a decreasing function, and

$$F^0(x) = F(c_n\,|x|^n) \simeq \left(\chi_{\{x:|x|\leq 1\}}(x) + |x|^{2-n}\,\chi_{\{x:|x|>1\}}(x)\right) \in L^{\frac{n}{n-2},\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

we get that $\chi_{[0,1]}^\circ \in D_{I_2}(X(\mathbb{R}^n))$.

Another argument to prove this part is the following:

Since, if $n \geq 3$ (see [2, Theorem 4.18, p. 228])

$$I_2 : L^1(\mathbb{R}^n) \to L^{\frac{n}{n-2},\infty}(\mathbb{R}^n) \text{ and } I_2 : L^{\frac{n}{2},1}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

is bounded, we have that

$$I_2 : (L^1(\mathbb{R}^n) \cap L^{\frac{n}{2},1}(\mathbb{R}^n)) \to (L^{\frac{n}{n-2},\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \subset X(\mathbb{R}^n)$$

is bounded, and hence $L^1(\mathbb{R}^n) \cap L^{\frac{n}{2},1}(\mathbb{R}^n) \subset D_{I_2}(X(\mathbb{R}^n))$.

$(5 \to 1)$ Since $n \geq 3$, we can use the classical formula of potential theory (see [10, p. 126])

$$-h = \triangle(I_2 h)$$

to conclude that there is a positive function $f = I_2\chi_{[0,1]}^\circ \in X(\mathbb{R}^n)$. Then $0 \leq f_t = I_2(\chi_{[0,1]}^\circ * \varphi_t) \in X(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ and $\triangle f_t \leq 0$. ■

We now consider particular examples, like the Lorentz spaces:

Corollary 3.6 *Let $1 \leq p < \infty$, and assume $\Lambda^p(\mathbb{R}^n, w)$ is a Banach space (i.e., $w \in B_p$ if $1 < p < \infty$ or $p \in B_{1,\infty}$ if $p = 1$, see [3]). Then, there exists a non-constant function $f \in \Lambda^p(\mathbb{R}^n, w)$ such that $\mathcal{M}(f) = f$ if and only if $n \geq 3$ and*

$$\int_1^\infty \frac{w(t)}{t^{p(1-2/n)}}\, dt < \infty.$$

In particular, this condition always holds, for $p > 1$ and n large enough.

Proof. The integrability condition follows by using the previous theorem. Now, if $w \in B_p$, then there exists an $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$, and hence, it suffices to take $n > 2/\varepsilon$. Observe that if $w = 1$ and $p = 1$, then $\Lambda^1(\mathbb{R}^n, w) = L^1(\mathbb{R}^n)$, which does not have the fixed point property for any dimension n. ■

Corollary 3.7 *Let $1 \leq p, q \leq \infty$ (if $p = 1$ we only consider $q = 1$). Then, there exists a non-constant function $f \in L^{p,q}(\mathbb{R}^n)$ such that $\mathcal{M}(f) = f$ if and only if $n \geq 3$ and*

$$\begin{cases} n/(n-2) < p \leq \infty \\ \qquad \textit{or} \\ p = n/(n-2) \ \textit{and} \ q = \infty. \end{cases}$$

Corollary 3.8 *(See [7]) Let $1 \leq p \leq \infty$. There exists a non-constant function $f \in L^p(\mathbb{R}^n)$ such that $\mathcal{M}(f) = f$ if and only if $n \geq 3$ and $n/(n-2) < p \leq \infty$.*

It is interesting to know when given an r.i. space $X(\mathbb{R}^n)$, the space $D_{I_2}(X(\mathbb{R}^n))$ is not trivial, or equivalently

$$\overline{D_{I_2}(X(\mathbb{R}^n))} := \left\{ f \in L^0([0,\infty)) : \left\| \int_t^\infty f^{**}(s) s^{2/n-1} ds \right\|_{\bar{X}(0,\infty)} < \infty \right\} \tag{2}$$

is not trivial. This will be done in terms of the fundamental indices of X. We start by computing the fundamental function of $D_{I_2}(X(\mathbb{R}^n))$.

Lemma 3.9 *Let X be an r.i. space on $\mathbb{R}^n$, $n \geq 3$. Let Y be given by (2). Then*

$$\phi_Y(s) \simeq s^{n/2} \left\| P_{1-2/n} \chi_{[0,s]} \right\|_X$$

where $P_{1-2/n} f(t) = t^{2/n-1} \int_0^t f(s) s^{-2/n} ds$.

Proof.

$$\begin{aligned} s^{n/2} P_{1-2/n} \chi_{[0,s]}(t) &\simeq s^{n/2} (\chi_{[0,s]}(t) + \left(\frac{s}{t}\right)^{1-2/n} \chi_{[s,\infty)}(t)) \\ &\simeq \int_t^\infty \chi_{[0,s]}^{**}(r) r^{2/n-1} dr. \end{aligned}$$

■

Theorem 3.10 *Let X be an r.i. space on $\mathbb{R}^n$, $n \geq 3$. Let Y be given by (2). Then*

1. *If $\overline{\beta}_X < 1 - 2/n$, then $Y \neq \{0\}$.*
2. *If $Y \neq \{0\}$ then $\underline{\beta}_X \leq 1 - 2/n$.*

Proof. 1.) Let $\chi_r = \chi_{[0,r]}$. Then

$$P_{1-2/n}\chi_r(t) = \int_0^1 \chi_r(\xi t)\frac{d\xi}{\xi^{n/2}} \leq c\sum_{k=0}^{\infty} 2^{-k(1-n/2)}\chi_{2^k r}(t).$$

Thus

$$\left\|P_{1-2/n}\chi_r\right\|_X \leq c\sum_{k=0}^{\infty} 2^{-k(1-n/2)}\phi_X(2^k r) \leq c\phi_X(r)\sum_{k=0}^{\infty} 2^{-k(1-n/2)}M_X(2^k).$$

Let $\varepsilon > 0$ be such that $\overline{\beta}_X + \varepsilon < 1 - 2/n$. Then by the definition of $\overline{\beta}_X$ it follows readily that there is a constant $c > 0$ such that

$$M_X(2^k) \leq c2^{k(\overline{\beta}_X+\varepsilon)},$$

and hence

$$\sum_{k=0}^{\infty} 2^{-k(1-n/2)}M_X(2^k) \leq \sum_{k=0}^{\infty} 2^{-k(1-n/2-\overline{\beta}_X-\varepsilon)} < \infty,$$

which implies that $\chi_r \in Y$.

2.) Since $Y \neq \{0\}$ if and only if $\left\|P_{1-2/n}\chi_{[0,1]}\right\|_X < \infty$ and

$$\sup_{t>0}\left(P_{1-2/n}\chi_{[0,1]}\right)^{**}(t)\phi_X(t) \leq \left\|P_{1-2/n}\chi_{[0,1]}\right\|_X < \infty, \tag{3}$$

and easy computations show that (3) implies that

$$1 \leq \sup_{t\geq 1}\frac{\phi_X(t)}{t^{1-2/n}} = c < \infty, \tag{4}$$

then, by (4)

$$\begin{aligned} M_X(a) &= \max\left(\sup_{t\geq 1/a}\frac{\phi_X(ta)}{\phi_X(t)}, \sup_{t<1/a}\frac{\phi_X(ta)}{\phi_X(t)}\right) \\ &= \max\left(\sup_{t\geq 1/a}\frac{\phi_X(ta)}{(at)^{1-2/n}}\frac{(at)^{1-2/n}}{\phi_X(t)}, \sup_{t<1/a}\frac{\phi_X(ta)}{\phi_X(t)}\right) \\ &\simeq \max\left(a^{1-2/n}\sup_{t\geq 1/a}\frac{t^{1-2/n}}{\phi_X(t)}, \sup_{t<1/a}\frac{\phi_X(ta)}{\phi_X(t)}\right). \end{aligned}$$

Thus, if $a < 1$, using again (4) we get

$$M_X(a) \geq a^{1-2/n}\sup_{t\geq 1/a}\frac{t^{1-2/n}}{\phi_X(t)} \geq a^{1-2/n}$$

which implies that

$$\underline{\beta}_X \leq 1 - 2/n.$$

■

Let us see that the converse in the previous theorem is not true.

Proposition 3.11 *There are rearrangement invariant spaces X such that*

1. $Y \neq \{0\}$ *and* $\overline{\beta}_X \geq 1 - 2/n$.
2. $Y = \{0\}$ *and* $\underline{\beta}_X < 1 - 2/n$.

Proof. Let $\varphi(t) = t^a \chi_{[0,1]}(t) + t^b \chi_{[1,\infty)}(t)$, with $0 \leq a, b \leq 1$. Let

$$X = \left\{ f \in L^0([0,\infty)) : \sup_{t>0} f^{**}(t)\varphi(t) < \infty \right\}.$$

Since φ is a quasi-concave function, we have that

$$\varphi(t) = \phi_X(t)$$

and

$$\underline{\beta}_X = \min(a,b),\ \overline{\beta}_X = \max(a,b).$$

On the other hand, the space Y defined by (2) is not trivial if and only if

$$b \leq 1 - 2/n.$$

Now, to prove 1) take $b \leq 1 - 2/n$ and $a \geq 1 - 2/n$. And to see 2) take $b > 1 - 2/n$ and $a \leq 1 - 2/n$. ■

Remark 3.12 If we consider

$$X_0 = \left\{ f \in L^0([0,\infty)) : \sup_{t>0} f^{**}(t) t^{1-2/n}(1 + \log^+ t) < \infty \right\}$$

and

$$X_1 = \left\{ f \in L^0([0,\infty)) : \sup_{t>0} f^{**}(t) \frac{t^{1-2/n}}{(1 + \log^+ t)} < \infty \right\}$$

then $\underline{\beta}_{X_i} = \overline{\beta}_{X_i} = 1 - 2/n$, $Y_0 = \{0\}$ and $Y_1 \neq \{0\}$.

Remark 3.13 It was proved in [7] that if we consider the strong maximal function (i.e, the maximal operator associated to centered intervals in $\mathbb{R}^n$), then there were no fixed points in any $L^p(\mathbb{R}^n)$ space, regardless of the dimension. The same argument works to show that $L^p(\mathbb{R}^n)$ cannot be replaced by any different r.i. space. Also, if we study this question for other kind of sets, like, e.g., Buseman–Feller differentiation bases (see [5]), then the only possible fixed points are the constant functions. This observation applies to any non-centered maximal operator (with respect to balls, cubes, etc.)

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Joaquim Martín
Dept. Mathematics
U. Autònoma de Barcelona
E-08193 Bellaterra (Barcelona), SPAIN *E-mail:* `jmartin@mat.uab.es`

Javier Soria
Dept. Appl. Math. and Analysis
University of Barcelona
E-08071 Barcelona, SPAIN *E-mail:* `soria@mat.ub.es`